Absolute and Conditional Convergence of an Alternating Series
The Basics

Finally we have an important concept, that of the absolute and conditional convergence of an alternating series. **This is very likely to be on any AP question that you see that concerns series, so make sure you spend enough time on it**.

A series will **converge absolutely** if the corresponding series of its absolute values converges. This is important because we have a lot of tests we can use to test the convergence of series with positive terms, so sometimes it is easier to show the positive term series will converge. **If a series converges absolutely, then both the original alternating series and the series that only contains positive terms will converge.**

The **Absolute Convergence Theorem** says that if the series of absolute values converges, then the alternating series will also converge. An alternating series **converges conditionally** when it does not converge absolutely, but the alternating series does converge (as shown with the Alternating Series Test).

Note – When you are given an alternating series, **you don't always have to check for absolute convergence.** You only have to do that when they ask you to.

1 **-** If the directions just say "determine if this alternating series converges", then you can go straight to the AST.

2**-** If the directions say "determine if the series converges absolutely or conditionally" then you have to use one of our other tests and look to see if the all-positive-terms series will converge. If it does converge, then you can say that the alternating series converges absolutely. If it does not converge, then go to the AST to see if the alternating series will converge conditionally.

Let's look at some examples: See below!

F: Determine if the series $\sum_{n=1}^{\infty}(-1)^n\frac{\sin n}{n^2}$ converges absolutely, converges conditionally, or diverges.

$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n}{n^2}$	This is our series
Consider the series $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$	Since we are asked to check for absolute convergence, look at the all-positive series first.
$\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series where p = 2 > 1 $\frac{\sin n}{n^2} \le \frac{1}{n^2}$ for all n > 1 Thus $\sum_{n=1}^{\infty}\frac{\sin n}{n^2}$ converges by the Direct Comparison Test.	I am going to use the Direct Comparison Test, although others would work. I notice that this series is similar to the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ and since sine is always between -1 and 1, my series will always less than or equal to the p-series.
$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n}{n^2}$ converges absolutely.	Our original series, then, must also converge.

Note – we never had to even use the AST here because we looked at the positive series first.

G: Determine if the series $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt[n]{10}\right)$ converges absolutely, converges conditionally, or diverges.

$\sum_{n=1}^{\infty}(-1)^n\left(\sqrt[n]{10}\right)$	This is our series.
Consider the series: $\sum_{n=1}^{\infty}\left(\sqrt[n]{10}\right)=\sum_{n=1}^{\infty}10^{1/n}$	Look at the positive series. When I rewrite it I can see that I can use the n^{th} term test.
$\lim_{n\to\infty}10^{1/n}=10^0=1$ The series diverges by the n^{th} term test. The series $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt[n]{10}\right)$ will also diverge by the n^{th} term test.	The terms approach zero do not approach zero, so we know it diverges. Since the first part of the AST is the n^{th} term test, we already know that the series will not converge conditionally either.

H: Determine if the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\ln n}$ converges absolutely, converges conditionally, or diverges.

$\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\ln n}$	This is our series.
Consider the series: $\sum_{n=1}^{\infty}\frac{1}{n\ln n}$	First I look at the absolute value series. I used the Integral Test

Let $f(x)=\dfrac{1}{x\ln x}$ The function is continuous, positive, and decreasing in value for all x ≥ 1 so the Integral Test applies. $\displaystyle\int_2^{\infty}\frac{1}{x\ln x}dx=\lim_{b\to\infty}\int_2^{b}\frac{1}{x\ln x}dx$ $\displaystyle\lim_{b\to\infty}\left[\ln(\ln x)\right]_2^b=\lim_{b\to\infty}\left[\ln(\ln b)-\ln(\ln 2)\right]=\infty$ The series diverges by the integral test.	to see that the absolute value of our series diverges Note: I had to use substitution to find the antiderivative: u = ln x Because this series diverges, we do not have absolute convergence.
$\displaystyle\lim_{n\to\infty}\frac{1}{n\ln n}=0$ the terms approach zero $\dfrac{1}{(n+1)\ln(n+1)}<\dfrac{1}{n\ln n}$ The terms decrease in size.	Now use the AST to determine if the alternating series will converge conditionally. Test to see if the terms approach zero and if they are decreasing.
The series converges conditionally by the Alternating Series Theorem.	State the conclusion.

I: Determine if the series $\displaystyle\sum_{n=1}^{\infty}(-5)^{-n}$ converges absolutely, converges conditionally, or diverges.

$\displaystyle\sum_{n=1}^{\infty}(-5)^{-n}=(-1)^{-n}(5)^{-n}$	This is our series. I rewrote it to see the factor that makes it an alternating series.
Consider the series $\displaystyle\sum_{n=1}^{\infty}(5)^{-n}=\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^n$ This is a convergent geometric series with $\lvert r\rvert = 1/5 < 1$	Consider the absolute value series and test it for convergence. It was nice to have such an easy geometric series!
The series converges absolutely.	State the conclusion.